December 1965

M-77-65-8

Technical Report

JUNCTURE STRESS FIELDS IN MULTICELLULAR SHELL STRUCTURES

Final Report

Nine Volumes

Vol. VIII Buckling Analysis of Segmental Orthotropic Cylinders Under Non-Uniform Stress Distribution

by

B. O. Almroth

F. A. Brogan

E. V. Pittner

Contract NAS 8-11480 to National Aeronautics and Space Administration
George C. Marshall Space Flight Center, Huntsville, Alabama

FOREWORD

This report is the result of a study on the buckling of segmental orthotropic cylinders under non-uniform axial compression. Work on this study was performed by staff members of Lockheed Missiles & Space Company in cooperation with the George C. Marshall Space Flight Center of the National Aeronautics and Space Administration under Contract NAS 8-11480. Contract technical representative was H. Coldwater.

This volume is the eighth of a nine-volume final report of studies conducted by the department of Solid Mechanics, Aerospace Sciences Laboratory, Lockheed Missiles & Space Company. Project Manager was K. J. Forsberg; E. Y. W. Tsui was Technical Director and Professor D. O. Brush of the University of California at Davis was a Consultant for the work.

The nine volumes of the final report have the following titles:

Vol. I	Numerical Methods of Solving Large Matrices
Vol. II	Stresses and Deformations of Fixed-Edge Segmental Cylindrical Shells
Vol. III	Stresses and Deformations of Fixed-Edge Segmental Conical Shells
Vol. IV	Stresses and Deformations of Fixed-Edge Segmental Spherical Shells
Vol. V	Influence Coefficients of Segmental Shells
Vol. VI	Analysis of Multicellular Propellant Pressure Vessels by the Stiffness Method
Vol. VII	Buckling Analysis of Segmental Orthotropic Cylinders under Uniform Stress Distribution
Vol. VIII	Buckling Analysis of Segmental Orthotropic Cylinders under Non-uniform Stress Distribution
Vol. IX	Summary of Results and Recommendations

SUMMARY

This volume presents a solution to the buckling problem of orthotropic cylindrical panels under non-uniform axial compression. For such a case the governing differential equations cannot be separated with respect to the space variables and thus analytical solutions are not available. The total potential energy of the system is expressed in terms of the displacement components and their derivatives. The energy formulation is transformed into a rational function of the displacements at the mesh points of a finite-difference net. The adjacent equilibrium theory of buckling is used to define the critical load as the eigenvalue of a large matrix.

Due to the fact that the finite-difference net has two dimensions, the degrees of freedom of the system is very large. Possible methods of solutions are therefore discussed with emphasis on computer economy.

CONTENTS

NOTATION

A	matrix
A_1; A_2	cross-sectional areas of stringer and ring
$b = R\Omega$	circumferential dimension of the panel
B	matrix
C	matrix
D_{X_i}; D_{T_j}	auxiliary functions
$\hat{e}_1$, $\hat{e}_2$	eccentricities of stringer and ring
E; E_1; E_2	moduli of elasticity of the shell, stringer, and ring
f	general notation for quantities in the energy expression [see Eq. (2.11)]
$(f)_i^j$; $(f_{,(\ )})_i^j$	quantities of f at the mesh point i, j and their derivatives
$\mathcal{F}_{(x,\theta)}$; $\mathcal{G}_{(x,\theta)}$	functions of x and θ
h	spacing between mesh points in axial direction
H	total number of mesh points along the generator
i, j	number of row and column in the mesh
J_1; J_2	torsional constancs for the cross-section of stringer and ring
k	spacing between mesh points along the curved edges
K	total number of mesh points along the curved edges
L	length of panel
M	matrix
$M_{(\ )}$; $N_{(\ )}$	nondimensionalized moments and stress resultants

R	radius of panel
r, s	spacing of rings and stringers
t	thickness of skin
U	nondimensionalized strain energy of the system
V	nondimensionalized total potential energy
u, v, w	nondimensionalized mid-surface displacement components in $\hat{w}$, $\hat{y}$, and $\hat{z}$ directions
$\hat{W}$	potential energy of external loads
$\hat{x}$, $\hat{y}$, $\hat{z}$	curvilinear coordinates for the mid-surface of the panel
X	vector of the displacement components u, v, w
Y	vector of constant terms
α_1, $\alpha_2 \ldots \alpha_7$	nondimensional geometrical parameters of the given structure
$\epsilon_{(\ )}$; $\gamma_{(\ )}$	direct and shear strains
$\chi_{(\ )}$	change of curvature or twist of mid-surface of panel
Ω	angle subtending the width of panel
$\theta = \hat{y}/R$	angular coordinate
$\gamma = \frac{1}{12}(\frac{t}{R})^2$	constant
λ	eigenvalue

Section 1
INTRODUCTION

When a cylindrical shell is subjected to an axially symmetrical load, the coefficients in the linear differential equations governing shell buckling are independent of the circumferential coordinate. Consequently the partial differential equations can be separated with respect to the space variables and the problem reduced to the solution of ordinary linear differential equations. If in addition prebuckling stresses are independent of the axial coordinate the governing differential equations have constant coefficients and an analytical solution is readily available. Analytical solutions can be obtained for cylindrical panels as well as complete cylinders, provided they are subjected to uniform longitudinal loading and simple support boundary conditions are chosen either for the curved or for the straight edges.

Buckling of the cylindrical panels of multicellular tanks or interstages was studied in Vol. 7 of this report. For simplicity and economy in computer analysis that investigation was restricted to panels subjected to uniform loading and the curved edges were assumed to be simply supported. The stress distribution in the cylindrical panels of multicellular structures, particularly in fuel tanks, will certainly not be uniform, but the analysis of Vol. 7 is still suitable for a study of the influence on the buckling load of such parameters as stiffener eccentricity, panel width, and web plate bending stiffness. Also it appears possible by use of engineering judgment to establish within reasonable limits an equivalent uniform stress distribution and thus use the results directly for design purposes.

The present analysis considers cylindrical panels with non-uniform stress distributions, and its primary purpose is to serve as an aid in the establishment of the equivalent uniform stress distribution. In this case the governing partial differential equations are not separable, and the numerical solution is somewhat unwieldy. Very few analyses of this type are available. For complete cylinders and a particularly simple case of non-uniform loading, Ref. 1 employs successfully double Fourier series. However, for the present case it is believed that numerical results can be more easily obtained by use of a two-dimensional finite-difference net. The number of mesh points needed depends on the load distribution as well as on the buckling pattern. Computers presently used have sufficient storage capacity and short enough instruction time that it may be practical to obtain solutions for complete cylinders when the loading varies smoothly over the surface and when the number of circumferential waves is moderate. For the relatively narrow panels, which are typical for multicellular structures, the situation is more favorable. It is expected that for stress distributions typical for multicellular structures the required computer time will be reasonable.

Section 2
THEORY

By use of finite-difference approximations for the derivatives it is, of course, possible to transform the governing partial differential equations and the boundary conditions into a set of linear simultaneous equations. However, it was decided here to use instead an energy approach in combination with finite-difference approximations. Such a procedure was for instance employed by Stein in Ref. 2. Its advantages are that no derivatives of higher order than second need to be approximated by finite differences, and that natural boundary conditions are automatically satisfied and thus need not be specified.

According to Vol. 7 we have for the strain energy of the skin, stringers, and rings:

$$U = \int_0^{L/R}\int_{-\Omega/2}^{\Omega/2} [\alpha_3\epsilon_x^2 + \alpha_4\epsilon_\theta^2 + 2\alpha_1\epsilon_x\chi_x + 2\alpha_2\epsilon_\theta\chi_\theta + 2\nu\epsilon_x\epsilon_\theta + \frac{1-\nu}{2}\gamma_{x\theta}^2$$

$$+ \alpha_5\chi_x^2 + \alpha_6\chi_\theta^2 + 2\nu\gamma^2\chi_x\chi_\theta + (\alpha_7 - 2\nu\gamma^2)\chi_{x\theta}^2]\, dx d\theta \qquad (2.1)$$

where

$$U = \frac{2(1-\nu^2)}{EtR^2}\hat{U} \quad ; \qquad \gamma^2 = \frac{1}{12}\left(\frac{t}{R}\right)^2$$

and

$$\alpha_1 = \left(\frac{\hat{e}_1}{R}\right)\left(\frac{E_1}{E}\right)\frac{A_1(1-\nu^2)}{st}$$

$$\alpha_2 = \left(\frac{\hat{e}_2}{R}\right)\left(\frac{E_2}{E}\right)\frac{A_2(1-\nu^2)}{rt}$$

$$\alpha_3 = 1 + \left(\frac{E_1}{E}\right)\frac{A_1(1-\nu^2)}{st}$$

$$\alpha_4 = 1 + \left(\frac{E_2}{E}\right)\frac{A_2(1-\nu^2)}{rt} \qquad (2.2)$$

$$\alpha_5 = \gamma^2 + (1 - \nu^2)\left(\frac{E_1}{E}\right) \frac{I_1 + A_1\hat{e}_2^2}{stR^2}$$

$$\alpha_6 = \gamma^2 + (1 - \nu^2)\left(\frac{E_2}{E}\right) \frac{I_2 + A_1\hat{e}_2^2}{rtR^2}$$

$$\alpha_7 = 2\gamma^2 + \frac{1 - \nu^2}{2tR^2}\left[\frac{J_1}{(1 + \nu_1)s}\left(\frac{E_1}{E}\right) + \frac{J_2}{(1 + \nu_2)r}\left(\frac{E_2}{E}\right)\right]$$

The strains ϵ_x, ϵ_θ, and $\gamma_{x\theta}$ in Eq. (2.1) represent the changes of strains in the middle-surface of the skin, which is due to deformation at buckling. Similarly χ_x, χ_θ, $\chi_{x\theta}$ represent the changes of curvature.

If the displacements of points in the middle surface in axial, tangential, and radial directions respectively are denoted by u, v, and w we have

$$\begin{aligned}
\epsilon_x &= u_{,x} + w_{,x}^2/2 \\
\epsilon_\theta &= v_{,\theta} + w + w_{,\theta}^2/2 \\
\gamma_{x\theta} &= u_{,\theta} + v_{,x} + w_{,x}\, w_{,\theta} \\
\chi_x &= -w_{,xx} \\
\chi_\theta &= -w_{,\theta\theta} \\
\chi_{x\theta} &= -w_{,x\theta}
\end{aligned} \tag{2.3}$$

By use of equations (2.1), (2.2) and (2.3) the total potential energy can be expressed in terms of the basic shell parameters and the displacement components.

Following classical linear stability theory, we assume that the displacements (and therefore also the stress resultants) consist of two parts. One part corresponds to the state at impending buckling, and the other represents infinitesimal increments from that state. Expressing the strain-displacement relations in terms of such prebuckling and incremental quantities and substituting into Eq. (2.1), we obtain the strain energy in the variational form:

$$U = U_o + \delta U + \frac{1}{2!}\delta^2 U + \frac{1}{3!}\delta^3 U + \frac{1}{4!}\delta^4 U \qquad (2.4)$$

Since the structure under load is in equilibrium at impending buckling, the first variation of the total potential energy must vanish:

$$\delta U = 0$$

Since the incremental displacements are infinitesimal, we may drop all incremental quantities higher than second order.

$$\delta^3 U = \delta^4 U = 0$$

Therefore the expression for the strain energy reduces to two terms:

$$U = U_o + \frac{1}{2!}\delta^2 U \qquad (2.5)$$

We assume that the effect of the prebuckling rotations on the magnitude of the buckling load is negligible. Thus the first variation of the strains and changes of curvature are expressed by the linear parts of Eqs. (2.3) only. Then the forms of the integrands in the two terms on the right side of Eq. (2.5) are very similar. The first contains prebuckling displacements only (subscript 0), and the second contains the same kind of terms with incremental displacements (subscript B) and in addition three mixed terms. Thus:

$$U_o = \int_o^{L/R} \int_{-\Omega/2}^{\Omega/2} \left[\mathcal{F}_{(x,\theta)}\right]_o \, dx \, d\theta \qquad (2.6)$$

$$\frac{1}{2}\delta^2 U = \int_o^{L/R} \int_{-\Omega/2}^{\Omega/2} \Big\{[\mathcal{F}_{(x,\theta)}]_B + \lambda[(N_x)_o(w_{,x})_B^2 + (N_\theta)_o(w_{,\theta})_B^2$$

$$+ 2(N_{x\theta})_o(w_{,x})_B(w_{,\theta})_B]\Big\} \, dx \, d\theta \qquad (2.7)$$

where:

$$\mathcal{F}_{(x,\theta)} = \alpha_3 u^2_{,x} + \alpha_4(v_{,\theta} + w)^2 + 2\nu u_{,x}(v_{,\theta} + w) + \frac{1-\nu}{2}(u_{,\theta} + v_{,x})^2$$

$$- 2\alpha_1 u_{,x} w_{,xx} - 2\alpha_2(v_{,\theta} + w)w_{,\theta\theta} + \alpha_5 w^2_{,xx} + \alpha_6 w^2_{,\theta\theta}$$

$$+ 2\nu\gamma^2 w_{,xx} w_{,\theta\theta} + (\alpha_7 - 2\nu\gamma^2)\, w^2_{,x\theta} \qquad (2.8)$$

The quantities $(N_x)_o$, $(N_\theta)_o$, and $(N_{x\theta})_o$ represent membrane forces in the prebuckling state. Due to the linearity of the prebuckling analysis it is possible to let these membrane forces correspond to the distribution under a unit load and to determine the critical value of the multiplier λ .

The total potential energy of the loaded shell is the sum of the strain energy U and the potential energy of the external forces. In a buckling analysis the critical load is defined as the load at which the second variation of the total potential energy achieves a relative minimum value of zero. For the loading considered, the second variation of the total potential energy is equal to $1/2\ \delta^2 U$. Therefore the critical load is determined by minimization with respect to the incremental displacement components of the expression:

$$\frac{1}{2}\delta^2 U = \int_0^{L/R} \int_{-\Omega/2}^{\Omega/2} \left[\mathcal{F}_{(x,\theta)} + \lambda\left\{\mathcal{G}_{(x,\theta)}\right\}\right] dx\, d\theta \qquad (2.9)$$

where

$$\mathcal{G}_{(w,\theta)} = (N_x)_o w^2_{,x} + (N_\theta)_o w^2_{,\theta} + 2(N_{x\theta})_o w_{,x} w_{,\theta} \qquad (2.10)$$

When the prebuckling loads $(N_x)_o$, $(N_\theta)_o$, and $(N_{x\theta})_o$ are uniform, an analytical solution is readily available, as has been shown in Vol. 7.

When the prebuckling loads are not uniformly distributed the governing partial differential equations representing a minimum value of the functional in Eq. (2.9) are not separable. Therefore, a finite-difference mesh is introduced. The spacing between mesh points in the axial and circumferential directions are respectively,

$$h = \frac{L}{(H - 1)R} \quad \text{and} \quad k = \frac{b}{(K - 1)R} = \frac{\Omega}{K - 1}$$

where H is the total number of mesh points along the generator and K is the total number of mesh points along the curved edges. If subscripts indicate position in the axial and superscripts indicate position in tangential direction, quantities (f) occurring in the energy expression and their derivatives are substituted by

$$\begin{aligned}
f &= (f)_i^j \\
(f_{,x})_i^j &= \frac{1}{2h}\left(f_{i+1}^j - f_{i-1}^j\right) \\
(f_{,\theta})_i^j &= \frac{1}{2k}\left(f_i^{j+1} - f_i^{j-1}\right) \\
(f_{,xx})_i^j &= \frac{1}{h^2}\left(f_{i+1}^j - 2f_i^j + f_{i-1}^j\right) \\
(f_{,\theta\theta})_i^j &= \frac{1}{k^2}\left(f_i^{j+1} - 2f_i^j + f_i^{j-1}\right) \\
(f_{,x\theta})_i^j &= \frac{1}{4hk}\left(f_{i+1}^{j+1} - f_{i-1}^{j+1} - f_{i+1}^{j-1} + f_{i-1}^{j-1}\right)
\end{aligned} \tag{2.11}$$

Furthermore, the integrals are replaced by finite summations and the energy is expressed in terms of displacement components u, v, and w at the mesh points, as discussed in the following section.

For definition of the boundary conditions fictitious mesh points are defined outside the panel. The load distribution in the prebuckling analysis is symmetrical about the plane $\theta = 0$. Therefore only one-half of the panel needs to be considered but the buckling pattern can be either symmetrical or antisymmetrical about this plane. The boundary conditions at $\theta = 0$ are

For symmetry   For antisymmetry

$$\begin{aligned} N_{x\theta} &= 0 & \qquad M_\theta &= 0 \\ Q_\theta &= 0 & N_\theta &= 0 \\ w_{,\theta} &= 0 & u &= 0 \\ v &= 0 & w &= 0 \end{aligned} \tag{2.12}$$

In both cases the first two conditions are automatically satisfied in a buckling analysis based on the energy method and only the last two need to be enforced. At the panel edges the analysis here assumes simple support conditions. Hence:

as $x = 0$ and $x = L/R$ at $\theta = \pm\Omega/2$

$$\begin{aligned} N_x &= 0 & \qquad N_\theta &= 0 \\ M_x &= 0 & M_\theta &= 0 \\ w &= 0 & w &= 0 \\ v &= 0 & u &= 0 \end{aligned} \tag{2.13}$$

Here again the first two conditions in each of the two groups are automatically satisfied. By use of boundary conditions specified here a number of the displacement components may be eliminated from the energy expression. The remaining values of the displacements at the meshpoints may be considered as the generalized coordinates of the system. With boundary conditions as specified by Eq. (2.13), the energy of the external forces vanishes and the total potential energy of the system reduces to the strain energy. Hence minimization of the strain energy with respect to the generalized coordinates signifies equilibrium. The minimization procedure leads to a set of homogeneous linear simultaneous equations. The lowest value of the load parameter, λ , which allows a non-trivial solution of this system corresponds to the critical load.

The quantities $(N_x)_o$, $(N_\theta)_o$, and $(N_{x\theta})_o$ in Eq. (2.7) could for instance be obtained from the analysis presented in Volumes II-VI. However, it would be impractical to read-in such a large quantity of data, and thus it is necessary to compute the prebuckling stress distribution internally. These computations will here be based on the energy approach.

The prebuckling membrane forces will be determined by use of the equilibrium equations, which can be derived from the total potential energy of the system. The energy consists of two parts, the strain energy as given by U_o in Eqs. (2.6) and (2.8) and the potential energy of the external forces. The loads on the edges of the panel are shown in Fig. 1. They are assumed to be symmetric about $\theta = 0$. At the curved edges the external loads are axial and at the straight edges only a shear force can be applied.

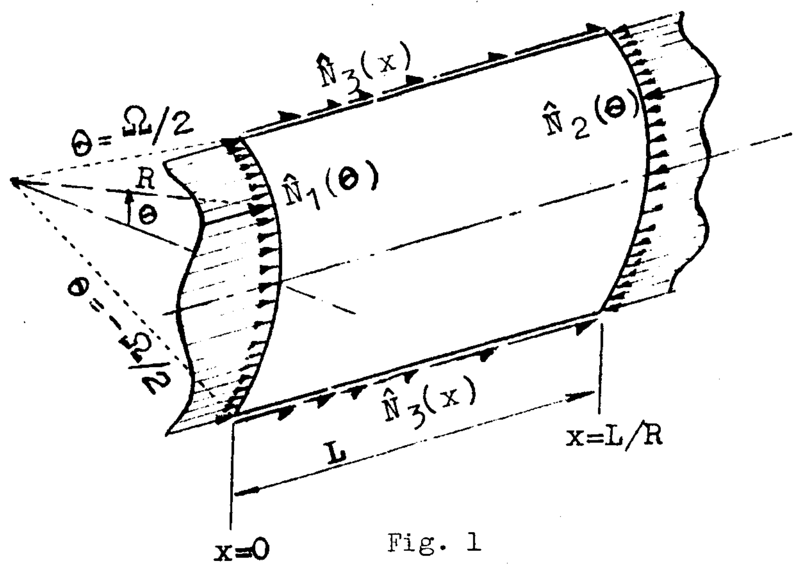

Fig. 1

Nondimensional load parameters are introduced such that

$$N_1(\theta) = \hat{N}_1(1 - \nu^2)/Et$$

$$N_2(\theta) = \hat{N}_2(1 - \nu^2)/Et$$

$$N_3(x) = \hat{N}_3(1 - \nu^2)/Et$$

If the potential energy is nondimensionalized in the same way as the strain energy

$$W = \hat{W}\frac{2(1 - \nu^2)}{EtR^2} = \int_{-\Omega/2}^{+\Omega/2} [u_1(\theta)\, N_1(\theta) + u_2(\theta)\, N_2(\theta)]\, d\theta + \int_0^{L/R} 2\, u_3(x)\, N_3(x)\, dx \qquad (2.14)$$

where u_1 , u_2 , and u_3 represent the axial displacements at $x = 0$, $x = L/R$, and $\theta = \pm\Omega/2$ respectively.

The finite-difference approximations which were employed in the buckling analysis are used here such that the total potential energy is given in terms of the displacement components at the mesh points. The boundary conditions are identical to those used in the buckling analysis except for the nonhomogeneous conditions which identify the edge loading.

$$
\begin{array}{llr}
\text{At } \theta = 0 & [(Q_\theta)_o = 0] & \\
& N_{x\theta} = 0 & \\
& w_{,\theta} = 0 & \\
& v = 0 & (2.15) \\
\text{At } \theta = \Omega/2 & [N_\theta = 0] & \\
& [M_\theta = 0] & \\
& w = 0 & \\
& \frac{1-\nu}{2}(u_{,\theta} + v_{,x}) = \hat{N}_3(x) & \\
\text{At } x = 0,\ L/R & [M_x = 0] & \\
& w = 0 & \\
& v = 0 & \\
& \alpha_3 u_{,x} = \begin{cases} \hat{N}_1(\theta) & \text{at } x = 0 \\ \hat{N}_2(\theta) & \text{at } x = L/R \end{cases} &
\end{array}
$$

In each of these groups the conditions within brackets correspond to minimum energy and will therefore be automatically satisfied. The external loads represented by $\hat{N}_1(\theta)$, $\hat{N}_2(\theta)$, and $\hat{N}_3(x)$ are not mutually independent but must satisfy the conditions of static equilibrium.

The boundary conditions and the condition of minimum energy constitute a linear equation system with the displacements at the mesh points as unknowns. After solution of this system the prebuckling membrane forces in Eq. (2.9) can be found by use of the strain-displacement relations Eqs. (2.3) and Hooke's Law.

Section 3
METHOD OF NUMERICAL ANALYSIS

In connection with the energy formulation of the previous section, we introduced a two-dimensional mesh net of displacement components u , v , and w with mesh lines parallel to the axial and circumferential coordinates (x and θ). In an example below (Figure 2) one-half of a cylindrical panel is covered by a rectangular mesh with uniform spacings h and k in the x and θ directions.

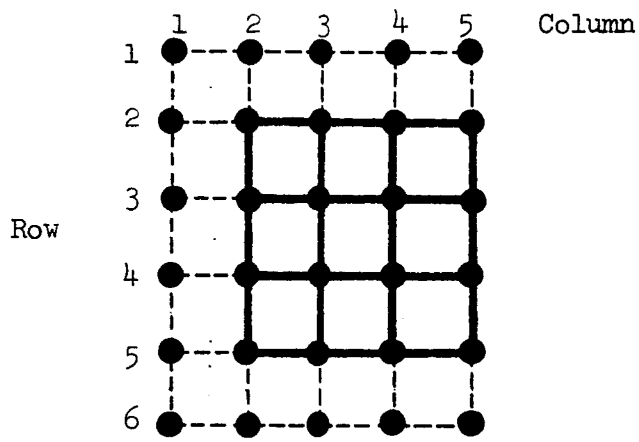

Fig. 2

In this example, rows 1 and 6 and column 1 consist entirely of fictitious points; rows 2 and 5 are the boundaries $x = 0$ and $x = L/R$ respectively; column 2 is the boundary $\theta = -\Omega/2$; column 5 represents a plane of symmetry or antisymmetry at $\theta = 0$.

Two auxiliary functions DX_i and DT_j are defined as follows:

$$DX_i = \begin{cases} 0 & \text{if } i \leq 1 \text{ or } i \geq H \\ 1/2 & \text{if } i = 2 \text{ or } i = H - 1 \\ 1 & \text{if } 2 < i < H - 1 \end{cases}$$

$$DT_j = \begin{cases} 0 & \text{if } j \leq 1 \text{ or } j \geq K + 1 \\ 1/2 & \text{if } j = 2 \text{ or } j = K \\ 1 & \text{if } 2 < j < K \end{cases} \qquad (3.1)$$

There H and K are the total numbers of rows and columns respectively in the mesh covering.

An approximation of the energy is obtained through replacement of the integration in Eq. (2.9) by finite summations:

$$\overline{U} = hk \sum_{i=2}^{H-1} \sum_{j=2}^{K} DX_i \cdot DT_j (\mathcal{F}_{(x_i\ \theta_j)} - \lambda \mathcal{G}_{(x_i\ \theta_j)}) \tag{3.2}$$

The values of $\mathcal{F}$ and $\mathcal{G}$ in terms of the displacement components u , v , and w are determined from the finite difference Eq. (2.11). Hence $\overline{U}$ becomes a rational function of the discrete variables u_i^j , v_i^j and w_i^j (see Eq. 2.12). Minimization of $\overline{U}$ then leads to the linear algebraic system:

$$\begin{aligned} \frac{\partial \overline{U}}{\partial u_i^j} &= 0 \qquad\qquad i = 1, H \\ \frac{\partial \overline{U}}{\partial v_i^j} &= 0 \qquad\qquad j = 1, K \\ \frac{\partial \overline{U}}{\partial w_i^j} &= 0 \end{aligned} \tag{3.3}$$

Rearrangement of the finite summation of Eq. (3.2) gives

$$\overline{U} = hk(U_1 - \lambda U_2)$$

where

$$\begin{aligned} U_1 &= \sum_{i=2}^{H-1} \sum_{j=2}^{K} DX_i DT_j\, \mathcal{F}_{(x_i, \theta_j)} \\ U_2 &= \sum_{i=2}^{H-1} \sum_{j=2}^{K} DX_i DT_j\, \mathcal{G}_{(x_i, \theta_j)} \end{aligned} \tag{3.4}$$

After a straightforward but lengthy calculation, the equations (3.3) can be defined and are shown in Table I.

In order to utilize the iterative methods described in Vol. I, a two-line ordering of the dependent variables u , v , and w is adopted. The order of the mesh stations is shown in Fig. 3.

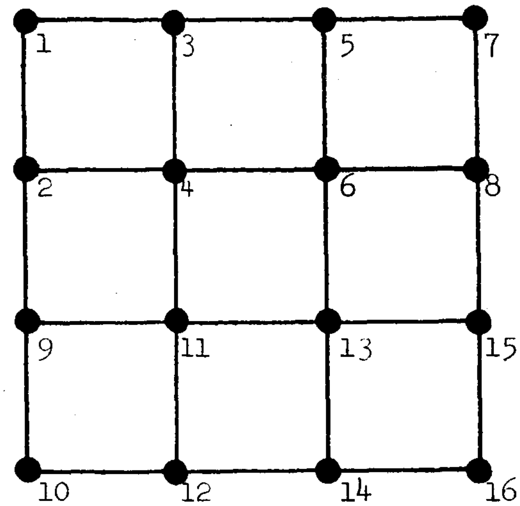

Fig. 3

The three unknown displacement variables at each mesh station are assigned successive positions in the overall system. The totality of displacement unknowns in each successive pair of mesh rows then constitutes a block of unknowns. When the expressions $\partial U_1/\partial u_i^j$, $\partial U_1/\partial v_i^j$, $\partial U_1/\partial w_i^j$ are listed in the same order, a banded matrix A in block tri-diagonal form is obtained. In a similar manner, the expressions $\partial U_2/\partial u_i^j$, $\partial U_2/\partial v_i^j$, $\partial U_2/\partial w_i^j$ lead to a rather sparse matrix B with the same form as A .

The displacement boundary conditions required for the buckling problem can be introduced without difficulty resulting in a slight modification of the A and B matrices. The linear system (3.3) can now be expressed as a matrix eigenvalue problem

$$(A - \lambda B)X = 0 \qquad (3.5)$$

where X is a vector of displacement components u , v and w . The least value of λ for which Eqs. (3.5) can be satisfied corresponds to the critical load.

The solution of Eq. (3.5) poses great difficulties when a large number of equations are involved. Preferred methods for the solution of the general eigenvalue problem of symmetric matrices, such as the Givens or Householder reductions (Ref. 3), are not feasible in this case. The computational effort required is prohibitive, primarily because the original banded form of the matrices is not retained throughout the computation. Fortunately, only the minimum eigenvalue of (3.5) is required for the buckling problem. Under these circumstances there are two approaches which can be considered feasible:

Define the function $f(z)$ by $f(z) = \mathrm{Det}\,(A - zB)$

Since the eigenvalues of (3.5) are the zeros of $f(z)$, iterative techniques for obtaining the roots of functions can be applied. The main computational effort is the repeated calculation of the determinant of a large matrix. Such methods are often useful for numerical exploration of the eigenvalue distribution and can be used to obtain particular eigenvalues. However, unless close estimates of the eigenvalues are available, the cost may be excessive.

The power iteration (Ref. 4), on the other hand, can be readily adapted to obtain the minimum eigenvalue of (3.5). Furthermore, the computations can be carried out in a manner which takes advantage of the sparseness of the A and B matrices. The power iteration for any matrix M is defined by

$$\begin{aligned} X_1 &= MX_o \\ X_{i+1} &= MX_i \end{aligned} \tag{3.6}$$

The iterates can also be written in the form

$$X_i = M^i X_o$$

Based on the properties of high powers of matrices, it can be shown that the iteration (3.6) converges to the dominant eigenvector of M when the corresponding eigenvalue is real and not multiple. This principle can be used to obtain an iteration which converges to the minimum eigenvector of (3.5). Suppose

$$AX = \lambda BX$$

Then if A is non-singular,

$$A^{-1}BX = 1/\lambda X \qquad (3.7)$$

Thus X is an eigenvector of the matrix $A^{-1}B$ and the eigenvalues of (3.7) are the reciprocals of the eigenvalues of (3.5). The power iteration can therefore be applied to obtain the dominant eigenvector of $A^{-1}B$ which corresponds to the minimum eigenvalue of (3.5). The resulting iteration is defined by

$$\begin{aligned} AX_1 &= BX_o \\ AX_{i+1} &= BX_i \end{aligned} \qquad (3.8)$$

Each step in this iteration requires the solution of a large linear system, $AX = Y$. The two-line cyclic iterative methods described in Volume I, Section 3 can be efficiently applied to this problem. A significant acceleration of the power iterations (3.8) can often be achieved by a shift of the eigenvalues.

Let $AX = \lambda BX$, and let p be a scalar. Then

$$(A - pB)X = AX - pBX = (\lambda - p)BX$$

$$\text{or} \qquad (A - pB)X = \mu BX \qquad (3.9)$$

where

$$\mu = \lambda - p$$

Thus the eigenvectors of (3.9) are the same as those of (3.8) and the eigenvalues μ have simply undergone a translation by the scalar p.

The corresponding power iteration now takes the form

$$(A - pB)X_1 = BX_o$$
$$(A - pB)X_{i+1} = BX_i \tag{3.10}$$

If p is less than but reasonably close to the lowest eigenvalue, λ, convergence may be attained in two or three iterations.

The solution of the prebuckling stress distribution will also be obtained by use of a two-dimensional finite difference mesh. Fortunately much of the numerical analysis of this problem can be based on the mathematical apparatus presented in connection with the buckling problem.

The potential energy of a cylindrical panel consists of two parts, the internal strain energy given in Eqs. (2.6)-(2.8) and the potential energy of the external forces given in Eq. (2.14). An approximation for the strain energy U_o is already available from Eq. (3.4)

$$U_o = hkU_1 = hk \sum_{i=2}^{H-1} \sum_{j=2}^{K} DX_i \, DT_j \, \mathcal{F}(x_i, \theta_j)$$

In a similar manner, an approximation for the potential energy of the external forces, W, can be obtained by use of finite-difference approximations and finite summations.

$$W = kW_1 + kW_2 + hW_3 \tag{3.11}$$

where

$$W_1 = \sum_{j=2}^{K} DT_j N_1(\theta_j) u_2^j$$

$$W_2 = \sum_{j=2}^{K} DT_j N_2(\theta_j) u_{H-1}^j \tag{3.12}$$

$$W_3 = \sum_{i=2}^{H-1} DX_i N_3(X_i) u_i^2$$

The minimization of the total energy V produces a linear equation system which must be solved for the unknown displacements

$$\frac{\partial V}{\partial u_i^j} = 0 \quad , \qquad \frac{\partial V}{\partial u_i^j} = 0 \quad , \qquad \frac{\partial V}{\partial w_i^j} = 0 \tag{3.13}$$

where

$$V = hkU_1 + kW_1 + kW_2 + hW_3$$

The expressions for the partial derivative of U_1 with respect to the displacement unknowns have already been given in Table I.

The complete system of linear equations for the prebuckling problem is obtained through addition of

$$\begin{aligned} \frac{\partial W_1}{\partial u_2^j} &= DT_j N_1(\theta_j) \\ \frac{\partial W_2}{\partial u_{H-1}^j} &= DT_j N_2(\theta_j) \\ \frac{\partial W_3}{\partial u_i^j} &= DX_i N_3(X_i) \end{aligned} \tag{3.14}$$

All other partials of W are zero. If the ordering of equations and unknowns described previously for the buckling problem is followed, we obtain a matrix equation

$$CX = Y \tag{3.15}$$

The matrix C is a banded block tri-diagonal matrix which differs from the matrix A of equation (3.5) only where modifications have been made to provide for the different boundary conditions.

Y is a vector of constant terms obtained primarily from the partial derivatives (3.14) but also including some terms derived from the non-homogeneous boundary conditions. The solution of (3.15) can be efficiently obtained by using a two-line cyclic iterative method as described in Volume I. It should be noted that little additional programming effort is required to solve the system (3.15). The subroutines used to solve the system $AX_{i+1} = BX_i$ for the buckling problem can be used with only minor changes to solve $CX = Y$ since C and A have identical forms and differ in only a few terms.

When the displacement components u, v and w have been found, the prebuckling stress distribution can be easily computed using finite difference approximations for the derivatives in the equations

$$
\begin{aligned}
N_x &= \alpha_3 u_{,x} + \nu(v_{2\theta} + w) - \alpha_1 w_{,xx} \\
N_\theta &= \alpha_4(v_{,\theta} + w) + \nu u_{,x} - \alpha_2 w_{,\theta\theta} \qquad (3.16) \\
N_{x\theta} &= \frac{1 - \nu}{2}(u_{,\theta} + v_{,x})
\end{aligned}
$$

$$\frac{\partial U_1}{\partial u_i^j} = -\frac{\alpha_3}{2h^2} DX_{i+1} DT_j u_{i+2}^j - \frac{\alpha_3}{2h^2} DX_{i-1} DT_j u_{i-2}^j$$

$$-\frac{1-\nu}{4k^2} DX_i DT_{j+1} u_i^{j+2} - \frac{1-\nu}{4k^2} DX_i DT_{j-1} u_i^{j-2}$$

$$+\left[\frac{\alpha_3}{2h^2}(DX_{i-1}+DX_{i+1})DT_j + \frac{1-\nu}{4k^2} DX_i (DT_{j-1}+DT_{j+1})\right] u_i^j$$

$$-\frac{1}{4hk}\left[2\nu DX_{i+1} DT_j + (1-\nu) DX_i DT_{j+1}\right] v_{i+1}^{j+1}$$

$$+\frac{1}{4hk}\left[2\nu DX_{i+1} DT_j + (1-\nu) DX_i DT_{j-1}\right] v_{i+1}^{j-1}$$

$$+\frac{1}{4hk}\left[2\nu DX_{i-1} DT_j + (1-\nu) DX_i DT_{j+1}\right] v_{i-1}^{j+1}$$

$$-\frac{1}{4hk}\left[2\nu DX_{i-1} DT_j + (1-\nu) DX_i DT_{j-1}\right] v_{i-1}^{j-1}$$

$$+\frac{\alpha_1}{h^3} DX_{i+1} DT_j w_{i+2}^j - \frac{\alpha_1}{h^3} DX_{i-1} DT_j w_{i-2}^j$$

$$-\left(\frac{2\alpha_1}{h^3} + \frac{\nu}{h}\right) DX_{i+1} DT_j w_{i+1}^j$$

$$+\left(\frac{2\alpha_1}{h^3} + \frac{\nu}{h}\right) DX_{i-1} DT_j w_{i-1}^j$$

$$+\frac{\alpha_1}{h^3}(DX_{i+1} - DX_{i-1}) DT_j w_i^j$$

$$
\begin{aligned}
\frac{\partial U_1}{\partial v_i^j} = &- \frac{(1-\nu)}{4h^2} DX_{i+1} DT_j v_{i+2}^j - \frac{(1-\nu)}{4h^2} DX_{i-1} DT_j v_{i-2}^j \\
&+ \left[\frac{(1-\nu)}{4h^2} (DX_{i+1} + DX_{i-1}) DT_j + \frac{\alpha_4}{2k^2} DX_i (DT_{j+1} + DT_{j-1}) \right] v_i^j \\
&- \frac{\alpha_4}{2k^2} DX_i DT_{j+1} v_i^{j+2} - \frac{\alpha_4}{2k^2} DX_i DT_{j-1} v_i^{j-2} \\
&- \frac{1}{4hk} \left[2\nu DX_i DT_{j+1} + (1-\nu) DX_{i+1} DT_j \right] u_{i+1}^{j+1} \\
&+ \frac{1}{4hk} \left[2\nu DX_i DT_{j-1} + (1-\nu) DX_{i+1} DT_j \right] u_{i+1}^{j-1} \\
&+ \frac{1}{4hk} \left[2\nu DX_i DT_{j+1} + (1-\nu) DX_{i-1} DT_j \right] u_{i-1}^{j+1} \\
&- \frac{1}{4hk} \left[2\nu DX_i DT_{j-1} + (1-\nu) DX_{i-1} DT_j \right] u_{i-1}^{j-1} \\
&- \left(\frac{2\alpha_2}{k^3} + \frac{\alpha_4}{k} \right) DX_i DT_{j+1} w_i^{j+1} \\
&+ \left(\frac{2\alpha_2}{k^3} + \frac{\alpha_4}{k} \right) DX_i DT_{j-1} w_i^{j-1} \\
&+ \frac{\alpha_2}{k^3} DX_i DT_{j+1} w_i^{j+2} - \frac{\alpha_2}{k^3} DX_i DT_{j-1} w_i^{j-2} \\
&+ \frac{\alpha_2}{k^3} DX_i (DT_{j+1} - DT_{j-1}) w_i^j
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial U_1}{\partial w_i^j} = {} & \frac{(\alpha_7 - 2\nu Y^2)}{8h^2k^2} DX_{i+1} DT_{j+1} w_{i+2}^{j+2} + \frac{(\alpha_7 - 2\nu Y^2)}{8h^2k^2} DX_{i+1} DT_{j-1} w_{i+2}^{j-2} \\
& + \frac{(\alpha_7 - 2\nu Y^2)}{8h^2k^2} DX_{i-1} DT_{j+1} w_{i-1}^{j+2} + \frac{(\alpha_7 - 2\nu Y^2)}{8h^2k^2} DX_{i-1} DT_{j-1} w_{i-2}^{j-2} \\
& + \left[\frac{2\alpha_5}{h^4} DX_{i+1} DT_j - \frac{(\alpha_7 - 2\nu Y^2)}{8h^2k^2} DX_{i+1} (DT_{j+1} + DT_{j-1}) \right] w_{i+2}^{j} \\
& + \frac{2\nu Y^2}{h^2k^2} (DX_i DT_{j+1} + DX_{i+1} DT_j) w_{i+1}^{j+1} \\
& + \frac{4\nu Y^2}{h^2k^2} (DX_i DT_{j-1} + DX_{i+1} DT_j) w_{i+1}^{j-1} \\
& + \frac{2\nu Y^2}{h^2k^2} (DX_i DT_{j+1} + DX_{i-1} DT_j) w_{i-1}^{j+1} \\
& + \frac{2\nu Y^2}{h^2k^2} (DX_i DT_{j-1} + DX_{i-1} DT_j) w_{i-1}^{j-1} \\
& - \left[\frac{4\alpha_5}{h^4} (DX_i + DX_{i+1}) DT_j + \frac{4\nu Y^2}{h^2k^2} (DX_i + DX_{i+1}) DT_j \right] w_{i+1}^{j} \\
& + \left[\frac{2\alpha_6}{k^4} DX_i DT_{j+1} - \frac{(\alpha_7 - 2\nu Y^2)}{8h^2k^2} (DX_{i+1} + DX_{i-1}) DT_{j+1} \right] w_i^{j+2} \\
& + \left[\frac{2\alpha_6}{k^4} DX_i DT_{j-1} - \frac{(\alpha_7 - 2\nu Y^2)}{8h^2k^2} (DX_{i+1} + DX_{i-1}) DT_j \right] w_i^{j-2} \\
& - \left(\frac{4\alpha_6}{k^4} + \frac{2\alpha_2}{k^2} + \frac{4\nu Y^2}{h^2k^2} \right) DX_i (DT_{j+1} + DT_j) w_i^{j+1} \\
& - \left(\frac{4\alpha_6}{k^4} + \frac{2\alpha_2}{k^2} + \frac{4\nu Y^2}{h^2k^2} \right) DX_i (DT_j + DT_{j-1}) w_i^{j-1} \\
& - \left[\frac{4\alpha_5}{h^4} (DX_i + DX_{i-1}) DT_j + \frac{4\nu Y^2}{h^2k^2} (DX_i + DX_{i-1}) DT_j \right] w_{i-1}^{j} \\
& + \left[\frac{2\alpha_5}{h^4} DX_{i-1} DT_j - \frac{(\alpha_7 - 2\nu Y^2)}{8h^2k^2} DX_{i-1} (DT_{j+1} + DT_{j-1}) \right] w_{i-2}^{j}
\end{aligned}
$$

$$+\left[\left(2\alpha_4+\frac{8\alpha_2}{k^2}+\frac{16\nu\gamma^2}{h^2k^2}\right)DX_i\,DT_j+\frac{2\alpha_5}{h^4}(DX_{i+1}+4DX_i+DX_{i-1})DT_j\right.$$

$$+\frac{(\alpha_7-2\nu\gamma^2)}{8h^2k^2}(DX_{i+1}+DX_{i-1})(DT_{j+1}+DT_{j-1})$$

$$\left.+\frac{2\alpha_6}{k^4}DX_i(DT_{j+1}+4DT_j+DT_{j-1})\right]w_i^j$$

$$+\left(\frac{2\alpha_2}{k^3}+\frac{\alpha_4}{k}\right)DX_i\,DT_j\,v_i^{j+1}-\left(\frac{2\alpha_2}{k^3}+\frac{\alpha_4}{k}\right)DX_i\,DT_j\,v_i^{j-1}$$

$$-\frac{\alpha_2}{k^3}DX_i\,DT_{j+1}\,v_i^{j+2}+\frac{\alpha_2}{k^3}DX_i(DT_{j+1}-DT_{j-1})v_i^j$$

$$+\frac{\alpha_2}{k^3}DX_i\,DT_{j-1}\,v_i^{j-2}-\frac{\alpha_1}{h^3}DX_{i+1}\,DT_j\,u_{i+2}^j$$

$$+\left(\frac{2\alpha_1}{h^3}+\frac{\nu}{h}\right)DX_i\,DT_j\,u_{i+1}^j+\frac{\alpha_1}{h^3}(DX_{i+1}-DX_{i-1})DT_j\,u_i^j$$

$$-\left(\frac{2\alpha_1}{h^3}+\frac{\nu}{h}\right)DX_i\,DT_j\,u_{i-1}^j+\frac{\alpha_1}{h^3}DX_{i-1}\,DT_j\,u_{i-2}^j$$

$$\frac{\partial U_2}{\partial u_i^j}=0$$

$$\frac{\partial U_2}{\partial v_i^j}=0$$

$$\frac{\partial U_2}{\partial w_i^j} = -\frac{N_{11}}{2h^2} DX_{i+1} DT_j w_{i+2}^j - \frac{N_{22}}{2k^2} DX_i DT_{j+1} w_i^{j+2}$$

$$- \frac{N_{11}}{2h^2} DX_{i-1} DT_j w_{i-2}^j - \frac{N_{22}}{2k^2} DX_i DT_{j-1} w_i^{j-2}$$

$$+ \left[\frac{N_{11}}{2h^2} (DX_{i+1} + DX_{i-1}) DT_j + \frac{N_{22}}{2k^2} DX_i (DT_{j+1} + DT_{j-1}) \right] w_i^j$$

$$- \frac{N_{21}}{2hk} (DX_{i+1} DT_j + DX_i DT_{j+1}) w_{i+1}^{j+1}$$

$$+ \frac{N_{21}}{2hk} (DX_{i+1} DT_j + DX_i DT_{j-1}) w_{i+1}^{j-1}$$

$$+ \frac{N_{21}}{2hk} (DX_i DT_{j+1} + DX_{i-1} DT_j) w_{i-1}^{j+1}$$

$$- \frac{N_{21}}{2hk} (DX_i DT_{j-1} + DX_{i-1} DT_j) w_{i-1}^{j-1}$$

REFERENCES

1. B. O. Almroth, "Buckling of a Cylindrical Shell Subjected to Nonuniform External Pressure," Journal of Applied Mechanics, Vol. 29, Dec 1962, pp 675-682

2. M. Stein, "The Influence of Prebuckling Deformations and Stresses on the Buckling of Perfect Cylinders," NASA TR R-190, 1964

3. A. S. Householder, "The Theory of Matrices in Numerical Analysis," Blaisdell Publishing Co., 1964

4. J. H. Wilkinson, "The Algebraic Eigenvalue Problem," Oxford University Press, 1965